

Worm Structure in Modified Swift-Hohenberg Equation for Electroconvection

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A theoretical model for studying pattern formation in electroconvection is proposed in the form of a modified Swift-Hohenberg equation. A localized state is found in two dimension, in agreement with the experimentally observed “worm” state. The corresponding one dimensional model is also studied, and a novel stationary localized state due to nonadiabatic effect is found. The existence of the 1D localized state is shown to be responsible for the formation of the two dimensional “worm” state in our model.

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When a system is driven away from its equilibrium state, it often responds in forming regular patterns. Pattern formation phenomena is rather ubiquitous, occurring in many different physical, chemical and biological systems. One of the most carefully studied systems is the Rayleigh-Benard convection (RBC), where high precision experimental measurements can be carried out and compared quantitatively with theoretical studies. The careful study of this rich physical system for the past twenty years has greatly advanced our understanding of systems far away from equilibrium [1].

The study of localized structure in nonequilibrium system has received a great deal of attention since being observed experimentally in binary-mixture RBC [2]. Even though binary-mixture RBC is a highly dissipative system, the localized structures behave much like soliton in integrable systems. For example, in certain parameter range, they can pass through each other without changing their structures. On the theory side, Thual and Fauve [3] were the first to study the behavior of a subcritical complex Ginzburg-Landau equation and found that in certain parameter range, there are indeed localized pulse solutions. The basic ingredients for the existence of localized structure are: 1) there has to be linear bistability, which guarantees the local stability of the peak and the tail of the pulse; 2) the nonlinear dispersion (the complex part of the coefficients for the nonlinear terms) is needed to stabilize the front connecting the peak and the tail of the localized solution. Both of these requirements seem to be consistent with the experimental system, because the initial bifurcation in binary-mixture RBC is subcritical Hopf bifurcation. Much work has since been devoted along these lines to understand the details of the experimental results [4,5].

Most of the experimental results in binary-mixture RBC were obtained in quasi one dimension, i.e., in a thin annulus [6]. Further efforts to extend these findings to two dimensional system have not revealed any similar 2D localized state as in 1D except for some time dependent patchy structure [7] and some long time transients [8]. Recently, M. Denning et al [9] studied electroconvection in nematic liquid crystal very carefully. Due to the anisotropic nature of the liquid crystal, they found that

the initial instability of the system is towards forming oblique rolls with certain angle with respect to the director of the nematic liquid crystal. They also found that the initial bifurcation is Hopf bifurcation [10]. Depending on the electrical conductivity, the pattern they observed above onset is either spatially extended time dependent state/spatial-temporal chaos state (STC) or some isolated localized state, which they named the “worm” state. The worm state is localized in the direction perpendicular to the director of the liquid crystal, but is extensive in the parallel direction. The worm state can move in the parallel direction. The internal structure of the worm state seems to consist of both orientations of the oblique rolls, and internal roll structure is moving relative to the motion of its envelop. It is the goal of this paper to understand the interesting structure of the worm state.

There are a few well established methods for studying pattern formation. Notably among them is the amplitude equation formalism which was used in [3] and related works [5,4] to study the pulse pattern in binary-mixture RBC. Amplitude equation describes the large scale and long time behavior of the envelop of the pattern. It is perturbative in nature and describes the system accurately at parameters close to the onset. The shortcoming of the amplitude equation is thus quite obvious. In writing down the amplitude equation, one has already broken the full spatial symmetry of the original system which might be important in 2D. In the case of subcritical bifurcation, the amplitude equation can only give qualitative results because the amplitude does not scale with the small parameter (reduced Rayleigh number). For the problem at hand, there are even more severe limitation for the amplitude equation. Since the spatial extension of the worm state in the perpendicular direction is only about a couple of wavelengths, there is no separation of length scales to justify the use of the amplitude equation.

To understand the experiments qualitatively, a phenomenological model is often useful. The Swift-Hohenberg (SH) equation [11] is an order parameter equation with the full symmetry of the original problem, and its linear properties agree with that of the original problem. The SH equation is phenomenological in nature, and usually can not be derived from the original equation.

tions. For different experimental systems, The Swift-Hohenberg equation can be modified in different ways from its original form to accommodate different physical situations and symmetry requirements, e.g., the Non-Bousinesq effect, mean flow effect [12], Hopf bifurcation [13,14] and rotating convection [15]. Because of its versatility and simplicity, the modified Swift-Hohenberg equation (MSHE) has become instrumental in understanding many pattern forming systems.

In writing down the modified Swift-Hohenberg equation for electroconvection, we know that the equation has to be anisotropic even at the linear level, and the equation has to be complex because the initial bifurcation is Hopf bifurcation. Let $\phi(\vec{x}, t)$ be the complex order parameter, we can write the order parameter equation as:

$$\begin{aligned} \partial\phi/\partial t = & (\epsilon + i\omega)\phi - \sigma((\partial_x^2 + q_x^2)^2 + b(\partial_x^2 + q_x^2)(\partial_y^2 + q_y^2) \\ & + (\partial_y^2 + q_y^2)^2)\phi + iv_g((\partial_x^2 + q_x^2) + a(\partial_y^2 + q_y^2))\phi \\ & + g_0|\phi|^2\phi + g_1|\phi|^4\phi \end{aligned} \quad (1)$$

ϵ is the reduced Rayleigh number, ω is the Hopf frequency, $\vec{q} = (q_x, q_y) = (\cos\theta, \sin\theta)$ is the linearly most unstable wavevector, b is an anisotropic parameter with the constrain $|b| \leq 2$, and σ is a complex constant. The first two lines on the RHS of eq. (1) represent the linear properties of the electroconvection system. It is easy to see that the system is most unstable at $|k_x| = q_x$ and $|k_y| = q_y$ for $\phi \sim \exp(ik_x x + ik_y y)$. v_g is proportional to the group velocity and a is another anisotropic parameter. When $a = 1$, the group velocity is along the wavevector direction \hat{q} . The last line on the RHS of eq. (1) contains the nonlinear coupling terms with complex coefficients g_0 and g_1 . In principle, the nonlinear terms can also be anisotropic, we only include the simplest terms possible here.

Since it is the goal of this paper to find the localized worm state, we focus our attention on the subcritical case where $\epsilon < 0$, $Re(g_0) > 0$ and $Re(g_1) < 0$. We can easily get rid of the $i\omega$ term in the linear part of the equation by a change of variable $\phi = e^{i\omega t}\psi$, so we will set $\omega = 0$ for now on. There are five real parameters: ϵ , a , b , θ and v_g and three complex parameters: σ , g_0 and g_1 for this model. We have numerically studied the MSHE extensively in the parameter space and identified certain parameter region where localized worm state is observed.

To demonstrate the existence of the worm state, we first show the behavior of the eq. (1) for a particular set of parameters: $\epsilon = -0.2$, $a = 1$, $b = 0$, $\theta = 23^\circ$, $v_g = 0.5$, $\sigma = 1.5$, $g_0 = 3 + i$ and $g_1 = -2.75 + i$. The equation is simulated in systems of size 64×64 , 128×64 and 256×64 with periodic boundary condition using both the second order finite difference method and spectral method with discretization $\Delta x = \Delta y = 0.5, 1.0$ and time step $\Delta t = 0.001, 0.01$. We start the system with random initial condition with large enough amplitude. The system quickly organizes itself into the worm like state. A snap shot of the 2D pattern for $Re(\phi(x, y))$ after the initial transient is shown in figure 1a.

A slice of the 2D pattern along the y direction at $x = 45$ is shown in figure 1b. The localization of the worm states in the y direction is apparent from fig. 1b. The Worm states travel in the x direction. From their length, the worm states in our simulation can be divided into two categories, which we call long worm and short worm. The length of the short worm does not change with time, and is usually ~ 20 , which is 3 basic wavelengths long. They travel in the x direction with constant velocity proportional to v_g . An example of a short worm can be seen near the bottom of fig.1a. The long worm's length grows with time and eventually extends over the whole length of the system because of the periodic boundary condition.

We have tested the sensitivity of the worm pattern to the parameters in our model. We find that there is a finite range of parameters where the worms appear. For example, if we change the value of ϵ while keeping the rest of the parameter unchanged, worm exists for $-0.10 > \epsilon > -0.25$. When ϵ is too small, there is no pattern; and when ϵ is too big, the pattern becomes extended instead. The worm state is quite insensitive to the values of a and b , as long as $a \sim 1$ and $|b| < 2$. For $b = 2$ and $a = 1$, the model becomes isotropic and the worm structure gives away to time dependent patchy structure [14]. The velocity v_g is important to give the worm a group velocity. The wavevector angle θ has to be small enough $\theta \leq 35^\circ$ to make the worm perfectly aligned in the x direction. There are also finite regions in the parameters g_0, g_1, σ where worm states are observed.

The worm states interact strongly with each other. When two short worms collide, they come out of the collision without changing their characteristics. When a short worm collides with a long worm, the short worm sometimes disappears. When two long worms approach each other off center, oblique rolls are excited in the region of their overlap until the worms pass through each other or one of the worms disappears. When two long worms collide head on, they stop each other and form a well defined boundary between them.

For the short worm, because of the spatial extension in both directions are about the same order, the formation of the short worm should be due to strong interaction between the two dimensions. However, for the long worms, due to the extensiveness of the worm in the x direction, we are able to separate the dependence in the two dimensions and therefore gain more understanding of the mechanism for the localization in the y direction. Indeed, Fourier analysis of the long worm along x direction shows that it is a good approximation to assume the x dependence to be a simple plane wave:

$$\phi(x, y, t) = \psi(y, t) \exp(ik_x x) \quad (2)$$

If we substitute the above ansatz into the original equation (1), we obtain a one dimensional dynamical equation for $\psi(y, t)$. For simplicity, we set $a = 1$ and $b = 0$:

$$\begin{aligned} \partial\psi/\partial t = & (\tilde{\epsilon} + i\tilde{\omega})\psi - \sigma(\partial_y^2 + q_y^2)^2\psi + iv_g(\partial_y^2 + q_y^2)\psi \\ & + g_0|\psi|^2\psi + g_1|\psi|^4\psi \end{aligned} \quad (3)$$

where, $\tilde{\epsilon} = \epsilon - \text{Re}(\sigma)(q_x^2 - k_x^2)^2$ and $\tilde{\omega} = \omega - \text{Im}(\sigma)(q_x^2 - k_x^2)^2 + v_g(q_x^2 - k_x^2)$. For the same reason as in two dimension, we can set $\tilde{\omega} = 0$.

We have studied the above 1D MSHE carefully. The numerical scheme is the same as in the two dimensional case, and we also started with a random initial condition with sufficient amplitude. In order to compare it to the two dimensional case, we have set the parameters $\sigma = 1.5$, $v_g = 0.5$, $g_0 = 3.0 + i$ and $g_1 = -2.75 + i$ to be the same as in the 2d calculation. We can vary $\tilde{\epsilon}$ because the value of k_x is undetermined a priori. For $q_y = \sin(\theta)$ with $\theta = 23^\circ$, we found a finite range of $\tilde{\epsilon}$ values, where localized structure is observed $-0.15 > \tilde{\epsilon} > -0.5$. A space-time plot of $\text{Re}(\psi(y, t))$ is shown in figure 2. It is clear that the system evolves to a final state with two localized pulses. Most remarkably, the pulses are not moving even in the presence of the group velocity term in eq. (3).

We found that the pulse solution can be written as:

$$\psi(y, t) = A(y) \exp(i\alpha(y, t)) \quad (4)$$

where the amplitude $A(y)$ is independent of time and is localized with a width of ~ 10 . Shifting the peak position to be at $y = 0$, the shape of the pulse is symmetric around $y=0$: $A(y) = A(-y)$. The phase of the pulse depends on time linearly:

$$\alpha(y, t) = \alpha_0(y) + \Omega t \quad (5)$$

with $\Omega = -0.25$. The shape of the time independent phase $\alpha_0(y)$ is depicted in figure 3(b). From figure 3(b), we see that the phase is symmetric around $y = 0$: $\alpha_0(y) = \alpha_0(-y)$. Away from the peak of the pulse, the phase is roughly linear in y :

$$\alpha_0(y) \sim k_y |y| + \text{const} \quad |y| > 5$$

with $k_y \sim 1$.

As we pointed out earlier in our paper, the existence of localized state in subcritical equation with complex coefficient was known [3]. However, a stationary localized pulse in the full equation including group velocity term is observed for the first time here. If one were able to eliminate the small scale structure and write the full equation in terms of the amplitude equation, one could use two amplitudes characterizing the left and the right moving wave packets. As shown in the work of Brand and Diessler [16], the left and the right moving pulses often can pass through each other without altering their own characteristics. Upon tuning the inter-coupling between the left and the right moving pulses, the pulses can form bound state which do not move in either directions. However, the structure of the bound state is such that the amplitudes of both the left and the right moving pulses are strongly suppressed in their coexisting region. In the middle of the bound state, both amplitudes are very small, and overall, within the bound state, both pulses still keep their own identities.

The pulse structure we observed here can not be explained as the bound state of the left and the right moving pulses in the amplitude equation because the localized state here does not have local minimum at the center. On the contrary, the amplitude is the maximum at the center of the localized pulse. Furthermore, in our simulation, we never observe any individual traveling pulse and the stationary localized structure always forms spontaneously as an whole object. The two halves of the pulse seem to have opposite phase velocity $v_p = \pm \Omega/k_y$. However, the periodicity in the phase $2\pi/k_y$ is much smaller than the linearly most unstable wavelength $2\pi/q_y$. In fact, the size of the pulse, i.e., the spatial extension of the whole pulse is smaller than $2\pi/q_y$. This clearly shows that the amplitude equation approach is invalid here. Therefore we believe that the localized pulse observed here is indeed a novel structure due to non-adiabatic effect, which can only be studied using MSHE where variation in all the length scales are kept.

According to the above analysis, the long worm can be understood as the combined structure of the 1d localized state in y direction and the simple harmonic behavior in x direction. In the y direction, the amplitude of the worm is maximum at the center and decays symmetrically as we move away from the center. The phases on the two sides have effective wavevectors $\vec{k}_\pm = (k_x, \pm k_y)$ which are different from the linearly most unstable modes $\vec{q}_\pm = (q_x, \pm q_y)$. The long worm expands while moving with the group velocity.

In summary, we have proposed a modified Swift Hohenberg model to count for the formation of localized worm state seen experimentally in electroconvection experiment. For a broad parameter range, we have found a solution of the MSHE which is localized in one direction and extended in the other. The structure of our solution resembles the experimental findings closely. The localization of the solution is further understood by simplifying the original two dimensional equation to 1D. A novel localized stationary pulse state is discovered in the 1D study which explains the worm structure.

To understand the experiments fully, we need to understand the dynamics of the worm in detail. For our current model, once a worm is formed, it will persist until it interacts with the boundary or another worm, while in the experiment, it seems that it can disappear by itself. Such a phenomena could be due to the strong external fluctuation or certain internal instability, which depends on the details of nonlinear terms. Another interesting issue is the interaction between worms, which also depends crucially on the form of the nonlinear terms. The MSHE can also be used to study the spatially extended STC state observed for larger electrical conductivity and the transition between the worm state and the STC state.

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FIG. 1. (a) Grey scale plot of $Re(\phi(x, y))$ showing localized worm structures. (b) The profile of the 2D pattern in (a) at $x=45$: $Re(\phi(45, y))$ versus y , the localized nature of the worm state is obvious.

FIG. 2. Space time plot of $Re(\psi(x, t)) + t/4$ versus x for time difference $dt = 4$

FIG. 3. (a) The amplitude of the 1D pulse $A(y)$ versus y ; (b) The stationary part of the phase of the 1D pulse $\alpha_0(y)$ versus y .









